Helping students conceptualize definition

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ABSTRACT: Definitions are core to science. Despite their importance and ubiquity many students have characteristic difficulties with reading, understanding and applying definitions. This contribution gives a categorized overview on the broad spectrum of these difficulties, describes teaching interventions designed to overcome these difficulties, and reports on the implementation of such interventions.

1 INTRODUCTION

While the concept of definition is central to science it is mathematics which most strongly relies on formal definitions. Mathematical texts and lectures necessarily introduce plenty of concepts even if they do not follow the infamous definition-theorem-proof style of presentation. A check of any mathematical textbook or lecture notes will reveal that any section easily introduces 5 to 10 new notions or concepts.

Naturally students are challenged by interiorising definitions. It turns out, however, that this is not only due to the content of the definition but also due to the concept of definition itself. There is research evidence on students' difficulties in understanding and applying definitions [1] and also on problematic views on the purpose of definitions in mathematics [2]. One can view definition as a threshold concept [3], i.e. a concept which once attained permits a new and previously inaccessible way of thinking about something. The challenging task of teaching then is to help students pass this threshold.

In this contribution I report on an ongoing reform of an introductory mathematics course for computer scientist. One of the reform efforts is to help students pass the described threshold with respect to definitions. The need to do so compellingly arose from the particular format the course is using: Just in Time Teaching [4], a teaching philosophy akin to flipped classroom with a strong focus on promoting conceptual understanding and diagnosing students' difficulties with subject matter. Various diagnosis instruments, in particular formative assessments, repeatedly and consistently show students' difficulties with definitions. These will be described in Section 2.

Based on the analysis of these difficulties I have implemented various activities for students in order to help them conceptualize definitions. These will be described in Section 3.

2 STUDENTS' UNDERSTANDING OF AND DIFFICULTIES WITH THE DEFINITION CONCEPT

Despite its ubiquity the concept of definition is surprisingly complex. Hence, it is not surprising that characteristic difficulties of students with this concept and even misconceptions are manifold and frequent. The following descriptions and categorization of difficulties related to definitions have been obtained by various means: literature review, classroom observations, semi-structured interviews of students, genetic decomposition [5], and analysis of students' work in formative assessment, exams, and the activities to be described in Section 3. Much of it is ongoing research. Details and research evidence will have to be published elsewhere.

2.1 Stipulatory nature of definitions

In order to understand students' difficulties with definitions it is helpful to contrast definitions in mathematics with those in encyclopaedias [2]. The latter describe the meaning of terms by reporting their usage. They explain terms. The former definitions typically intend to stipulate the usage of a term. They create terms. To provide an example: The mathematical definition of subset does not intend to describe how mathematicians use this term. It stipulates the usage of this term for a certain meaning. In fact, it intends to create the concept "subset" in the mind of the student. In a way, the definition pretends that neither the related concept nor term had existed before and asks the student to create them in his or her mind for further usage.

In fact, classroom observations and student surveys indicate that in reading mathematical texts students tend to skip definitions or to skim through them. Instead they prefer to head to the examples section of the text and try to distil the meaning of the defined concept from examples.

2.2 Psychological issues

The described behaviour of students can also be understood in terms of two contrasting processes of abstraction introduced by Piaget [6]. Definition and abstraction are linked as both are related to categorical terms. According to Piaget, empirical abstraction consists in creating a category by deriving common characteristics from a class of objects. This is largely based on perception, as opposed to reflection which is at the core of reflective abstraction which Piaget believed was the mental mechanism by which individuals construct mathematical concepts.

Distilling definitions from examples is akin to empirical abstraction. Whereas what students would need to do is to identify the operations (i.e. defining criteria) of the definitions, reflect on them, and by that internalize the defined notion.

2.3 Definitions as functions

Definitions can be conceptualized as mappings or functions. From that perspective definitions take any object as input and return true as an output if and only if the input object satisfies all defining criteria and, hence, can be termed by the notion introduced by the definition. Phrased in technical language, definitions can be viewed as predicates. This functional view aligns with the stipulatory nature of definitions in that the input to output mapping of functions can be arbitrary.

Again this view comes with its own challenges related to characteristic student difficulties with respect to functions [7] for instance the common misconception that functions need to map numbers to numbers.

2.4 Differentiation between definition and proximate concepts

Students often have difficulties telling apart definitions from theorems. From a perspective of formal logic both definitions and theorems are predicates. Definitions are predicates created via stipulation, theorems are predicates created via insight or argumentation. Both deal with properties. Definitions do so for the purpose of stipulation, theorems for the purpose of reasoning. Often texts add to these difficulties when defining concepts on the fly within a theorem about this concept thus blurring definition and theorem.

Students' difficulties with differentiating between these two concepts manifest themselves when they are not able to classify whether a statement is a definition, a theorem, or none of both. Also students' frequent misuse the phrase "is defined" for expressing that "something has the property" might indicate that they are not able to differentiate the two concepts. A typical example for such a misuse would be the statement "The empty set is defined to be a subset of any set" (which is not the case by definition but can be derived from the definition of subset and, hence, is a theorem).

2.5 Logical structure of definitions

Reading and understanding definitions requires some understanding of basic concepts of logic. Mathematical definitions state the necessary and sufficient conditions for an object to be named by the stipulated term. Quite often definitions are phrased as equivalences: Something is called such and such if and only if the stated requirements are met.

For many students, however, the meaning of biconditional statements (involving "if and only if") is not clear. They also might have difficulties to discriminate such statements from conditionals statements (involving "if … then"). In fact in everyday speech biconditional statements are often expressed via "if … then" and the biconditional meaning is inferred from context. A classic example is the sentence "Your mother says: If you finish your plate, you will have dessert." The context (encouraging children to empty their plate) implies that this statement is biconditional: If and only if you finish your plate, you will have dessert.

To make things worse, at least German textbooks tend to prefer if-then-phrases when defining concepts, e.g. "A matrix is singular, if its determinant is 0". Thus students need to infer the biconditional nature of the statement from the context which here is definition. That is, students need to know about the logical structure of definitions in order to decode words properly into their logical meaning. More often than not, however, texts lack the keyword "definition" requiring students to make use of additional contextual information such as phrases like "is called" or simply "is" (cf. the above definition of "singular") in order to identify the stipulatory nature of the statement.

Also definitions quite often come in the form of quantified statements, e.g. involving the keyword "for all" or its disguised forms ("each", "always" etc.). Students' characteristic difficulties with quantification are well documented in the literature [8] and naturally can be observed when requiring students to work with definitions.

2.6 Philosophical issues

Often conceptual difficulties can also be observed on a historical scale. The creation of many scientific concepts had been challenging for the scientific community and the related difficulties are often the same as or similar to those students face in their effort of internalising a concept.

What characterises a definition has been debated in philosophy and developed from a notion that emphasises the demarcation of concepts (hence the name definition) to various slightly different ways of stating conditions to be used to decide whether something falls into the defined category. From that perspective it is not surprising that students face problems in conceptualising definitions since the scientific community faced them as well.

3 INTERVENTIONS

As is so often the case with teaching conceptual understanding presentation in lectures or textbooks, however lucid, is likely to be insufficient. Students need to be given the opportunity to internalise concepts by working on meaningful activities involving these concepts. This section describes a number of teaching interventions that derive from the findings described in the previous section and that I found to be useful in an ongoing endeavour of helping students conceptualize definition.

3.1 Writing definitions as computer code

As analysed in Section 2.3 definitions can be viewed as predicates. A function implementing the definition of a given concept returns true if and only if the object given as input to this function satisfies the defining properties of this concept.

To give an example: In set theory the concept subset is usually defined in the following way: A set *A* is called a subset of set *B*, if and only if all elements of *A* or also elements of *B*. In a suitable programming language such as setlX (which is an evolution of setl [9]) this can be expressed as

```
isSubset := procedure(A,B){
                return forall (element in A | element in B);
                };
```

This code does nothing more than checking whether all elements of *A* or also elements of *B*. Hence, `isSubset({1,3},{1,2,3})` returns true as {1,3} is a subset of {1,2,3}, i.e. all elements of {1,3} are also elements of {1,2,3}. However, `isSubset({1,4},{1,2,3})` does not return true as {1,4} is not a subset of {1,2,3}, i.e. not all elements of {1,4} (notably 4) are elements of {1,2,3}. Likewise, `isSubset(2,{1,2,3})` does not return true as 2 is not a subset of {1,2,3} (it is not even a set).

Writing such snippets of code requires students to actually read definitions consciously and thus identifying the defining properties of a concept. It helps them to become aware of the stipulatory nature of definitions, and also to practise the language of logic which is at the heart of definitions not only in mathematics. Having implemented definitions as code also enables students to experiment with the code by letting the computer check whether given objects satisfy the defining properties of a concept. This is particularly helpful in limiting cases, such as the question whether the empty set is a subset of a given set.

Of course the code written by students needs to be checked and students need to receive feedback on it. At least checking for correctness can, however, by automated by software [10], thus considerably reducing the time demand on the instructor's side for implementing such interventions.

While such programming tasks prove to be very helpful they do not come without a price. They require students to learn a suitable declarative programming language and class time to be taken to introduce and practise programming in that language. Even for computer science students this can be challenging and often is not welcomed by many students. In my experience students' concerns can be sufficiently mitigated by taking them serious and by demonstrating that declarative programming languages such as setlX provide functionality that is not or not readily available with the imperative programming language they usually learn as a first language.

3.2 Eliciting students' nonconforming definitions

Students need help to understand the stipulatory character of definitions. A peer instruction [11] activity I have found to be useful for this purpose is to ask students in class whether $\sqrt{x^2}$ equals $\pm x$, $|x|$, or x. A poll on students' answers always results in a dissent which remains even after students discussed this with their neighbours. This helps them to realize that they need to stipulate the meaning of square root in order to avoid what they had just experienced: miscommunication in a learning community due to the fact that they individually had created different meanings of square root in their minds.

3.3 Classifying statements as definitions or theorems

As described in Sections 2.4 and 2.5 students often are not able to tell apart definitions from theorems if they do not come with the respective keyword. Asking students to classify statements into one of these two categories helps them to identify characteristics of definitions and theorems. Such activities are possibly even more effective when carried out in some sort of group activities which encourage students to show their thoughts and reasoning.

4 SUMMARY

Definition is a surprisingly difficult, troublesome, integrative, and transformative concept representing a threshold concept in mathematics if not the sciences in general. Various aspects ranging from philosophical to psychological issues add to the threshold character of definition. Based on research results of students' difficulties with definitions formative assessments and learning tasks can be designed to help student overcome these difficulties. One kind of learning tasks that seems to be particularly helpful consists of writing snippets of computer code. In a way these programming tasks can be viewed as "explaining specific definitions to the computer". They put the students into the role of teachers with the computer being their student.

It is interesting to note that on an abstract level what is common to most interventions advocated here is dialogue. This seems plausible not only on pedagogical grounds but also by the very stipulatory nature of definitions: Stipulation needs someone else to make an agreement with.

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